MATHEMATICS SPECIALIST UNIT 3

Section Two:
Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student number: In figures

   In words   _____

   Your name   _____

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

Consider the following system of equations, where a and b are constants.

$$x - 2y + z = 1$$
$$2x + 2y - z = 5$$
$$2x + ay + 2z = b$$

For each of the following cases, determine the number of solutions that exist for the system and briefly interpret the system geometrically.

(a)　　$a = 2, \ b = -4.$ (3 marks)

(b)　　$a = -4, \ b = -2.$ (3 marks)

Question 10   **(7 marks)**

(a)   Determine the values of the real constant p and the real constant q given that $z + 3 - 2i$ is a factor of $z^3 + pz + q$.

(4 marks)

(b)   Clearly show that $3 - i$ is a root of the equation $z^3 - 8z^2 + 22z - 20 = 0$.   (2 marks)

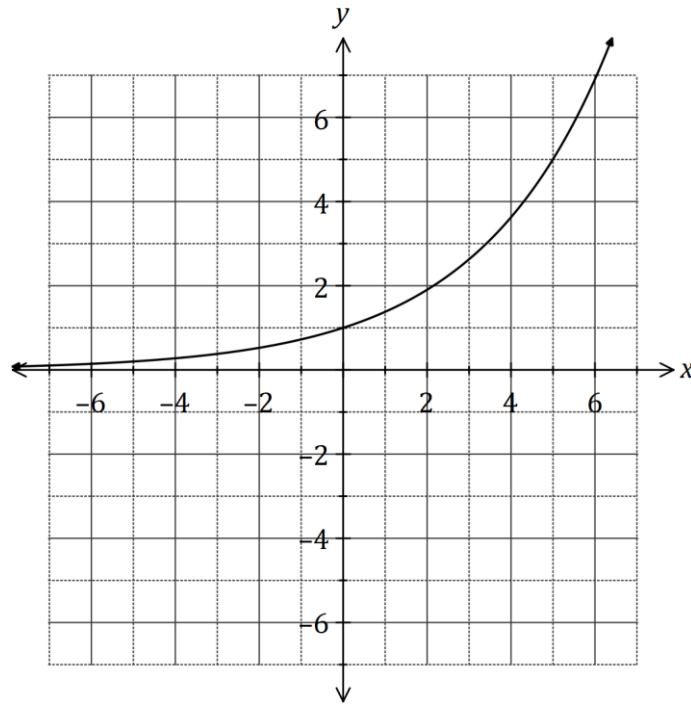(c)   State all three solutions of $z^3 - 8z^2 + 22z - 20 = 0$.   (1 mark)

Question 11   **(6 marks)**

(a)   Explain why the function $f(x) = \sin x$, where $x \in \mathbb{R}$, is not one-to-one.   (1 mark)

(b)   The graph of $y = g(x)$ is shown below. Sketch the graph of $y = g^{-1}(x)$ on the same axes.

(2 marks)

(c)   The inverse function of h is defined as $h^{-1}(x) = x^2 + 10x + 22$ for $x \leq -5$. Determine the defining rule for $h(x)$ and state its domain.   (3 marks)
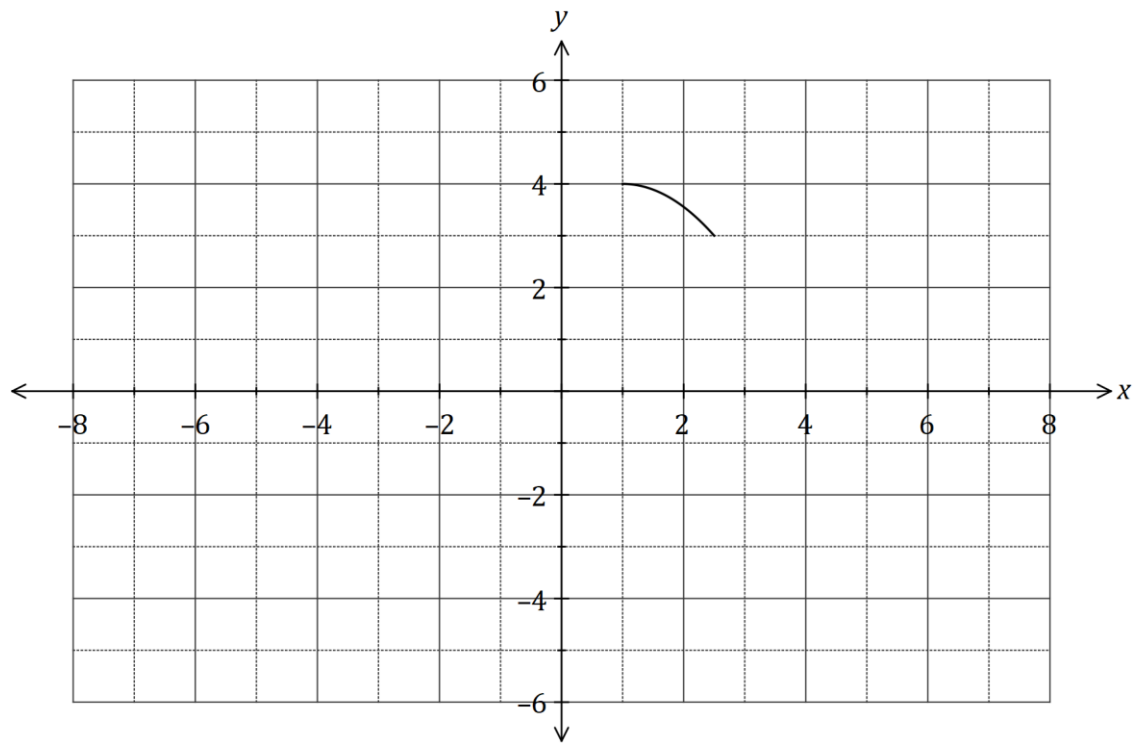
Question 12 **(9 marks)**

The position vector of a small body is $\mathbf{r}(t) = (1 - 3\sin(t))\mathbf{i} + (2 + 2\cos(2t))\mathbf{j}$ where t is the time in seconds since motion began.

(a)   Show that the body is stationary when $t = \frac{\pi}{2}$ and state its position at this time.   (3 marks)

(b)   Derive the Cartesian equation of the path of the body.   (4 marks)

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(c)   Complete the following plot to show the path taken by the body.   (2 marks)
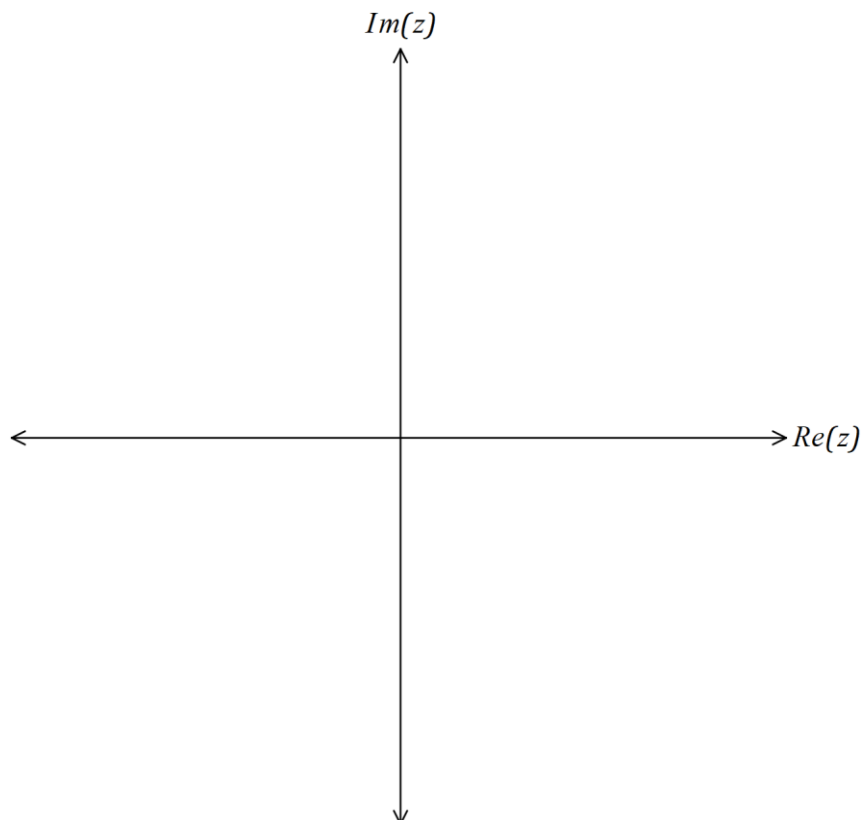
Question 13 (9 marks)

Let $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

(a) Express w, w^2, w^3 and w^4 in the form $r \operatorname{cis} \theta$, $-\pi < \theta \le \pi$. (2 marks)

(b) Sketch w, w^2, w^3 and w^4 as vectors on the Argand diagram below. (2 marks)

See next page

(c)   Describe the transformation in the complex plane of any point z when it is multiplied by w.

                                                                        (2 marks)

(d)   Simplify

            (i) $w + w^3 + w^5 + w^7$.   (1 mark)

            (ii) $w + w^3 + w^5 + \cdots + w^{2017} + w^{2019}$.   (2 marks)

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Question 14 **(8 marks)**

The position vectors of two particles at time t are given below, where a is a constant.

$$\mathbf{r}_A = 8\mathbf{i} - 5\mathbf{j} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \text{ and } \mathbf{r}_B = 3\mathbf{i} + a\mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

The paths of the particles cross at P but the particles do not meet.

(a) Determine the value of the constant a and the position vector of P. (5 marks)

(b) Show that the point $(1, -5, 4)$ lies in the plane containing the two lines. (3 marks)

Question 15   **(7 marks)**

(a)   Solve the equation $z^5 - 32i = 0$, writing your solutions in polar form $r \operatorname{cis} \theta$.   (4 marks)

(b)   Use your answers from (a) to show that $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0$.

                                                                  (3 marks)

See next page

Question 16   **(9 marks)**

A pole and a wall stand vertically on horizontal ground. A small projectile is launched from the pole at a height of 4.25 m above the ground and sometime later hits the wall at a height of 1.29 m above the ground. The projectile has an initial velocity of 28 ms^{-1} at an angle of $54°$ above the horizontal.

Any effects of air resistance and wind can be ignored. Let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors in the horizontal and vertical (upward) directions and the foot of the pole be at $(0, 0)$.

The acceleration acting on the projectile is given by $\mathbf{a}(t) = -9.8\mathbf{j}$ ms^{-2}.

(a)   Use the information above to derive vector equations for the velocity $\mathbf{v}(t)$ and displacement $\mathbf{r}(t)$ of the projectile at any time t.   (3 marks)

(b)   Determine

      (i)   the time that the projectile takes to travel between the pole and the wall.

                                              (2 marks)

See next page

    (ii)   the speed of the projectile at the instant it hits the wall.   (2 marks)

    (iii)   the length of the path taken by the projectile between the pole and the wall.
                                          (2 marks)

Question 17   **(8 marks)**

Sphere S has diameter PQ, where P and Q have coordinates $(6, -2, -3)$ and $(-2, 4, 1)$ respectively.

(a)   Determine the vector equation of the sphere.   (3 marks)

(b)   Show that the point $(5, 5, 2)$ lies outside the sphere.   (2 marks)

(c)   Show that the line with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ is tangential to the sphere.

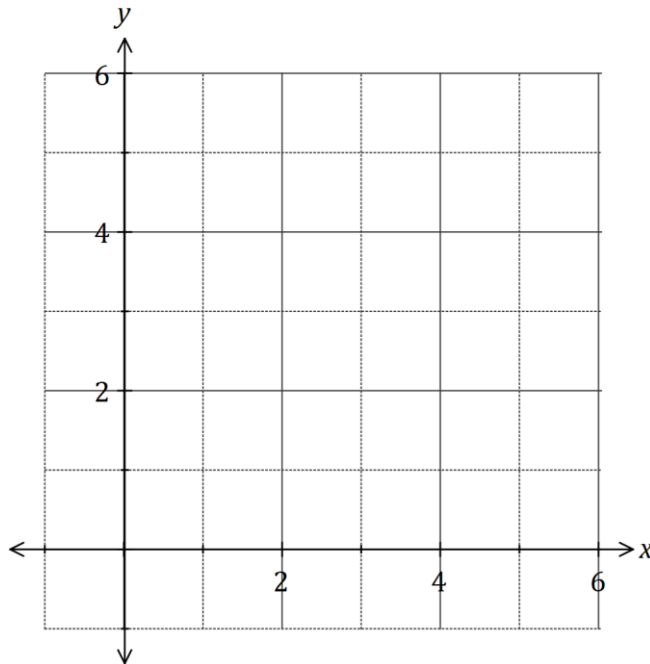   (3 marks)

Question 18   **(9 marks)**

Let $f(x) = \sqrt{x-2}$, $g(x) = \dfrac{6}{x}$ and $h(x) = f \circ g(x)$.

(a)   Determine an expression for $h(x)$ and show that the domain of $h(x)$ is $0 < x \leq 3$.

                                                           (3 marks)

(b)   Determine an expression for $h^{-1}(x)$, the inverse of $h(x)$.   (1 mark)

(c)   Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the axes below.   (3 marks)

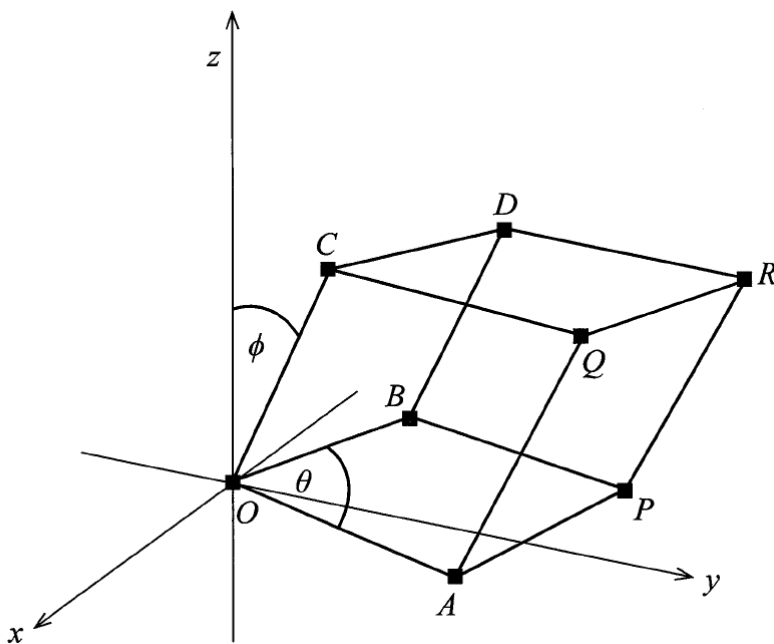(d)   Solve $h(x) = h^{-1}(x)$, correct to 0.01 where necessary.   (2 mark)

Question 19   **(6 marks)**

A parallelepiped is a prism with each face a parallelogram. Let $OAPB$ be the parallelogram formed by the horizontal sides $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ where

$$\underset{\sim}{a} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix}.$$

The third side that forms the parallelepiped is $\underset{\sim}{c} = \overrightarrow{OC}$ where $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

Let θ = the size of $\angle AOB$

ϕ = the angle between $\overrightarrow{OC}$ and the positive z-axis

(a)   Determine $\underset{\sim}{a} \times \underset{\sim}{b}$.   (2 marks)

The volume of any prism can be found by the formula *Volume = Area (base) × h,* where h is the perpendicular height of the prism.

It is also true that $\left| \underset{\sim}{a} \times \underset{\sim}{b} \right| = \left| \underset{\sim}{a} \right| \left| \underset{\sim}{b} \right| \sin \theta$.

(b)   Explain why $\underset{\sim}{c} \bullet \left(\underset{\sim}{a} \times \underset{\sim}{b} \right)$ will determine the volume of the parallelepiped.   (2 marks)

(c)   Hence determine the exact volume of the parallelepiped.   (2 marks)

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Question 20   **(8 marks)**
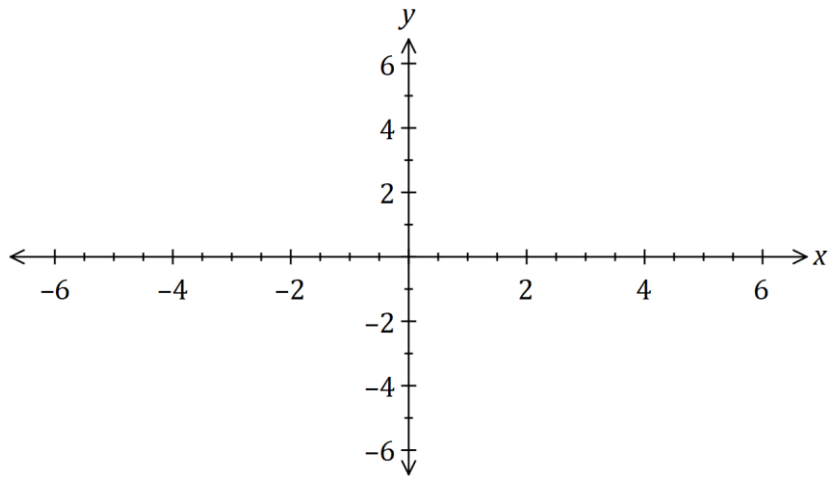
Let $f(x) = 6 - |3x - 6|$.
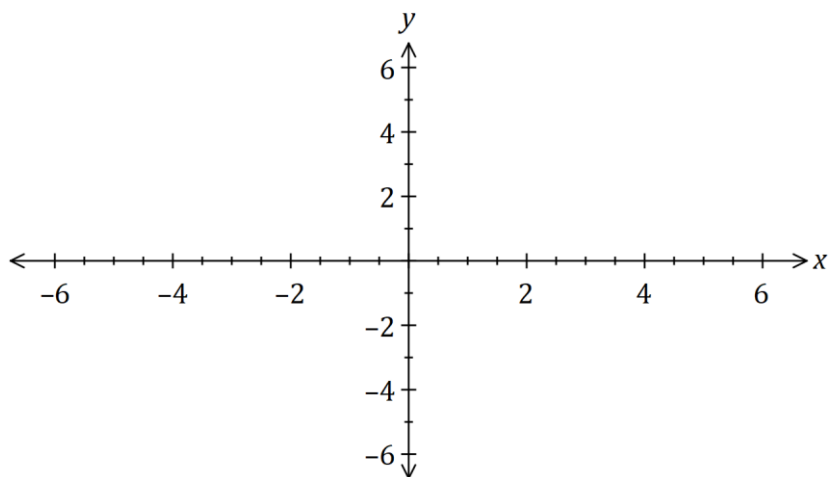
(a)   Sketch the graph of $y = f(x)$ on the axes below.   (2 marks)

(b)   Sketch the graph of $y = f(|x|)$ and hence solve $f(|x|) - 3 = 0$.   (3 marks)

(c)   The equation $f(x) = a|x + b| + c$ is true only for $-1 \leq x \leq 2$. Determine the value of each
       of the constants a, b and c.   (3 marks)
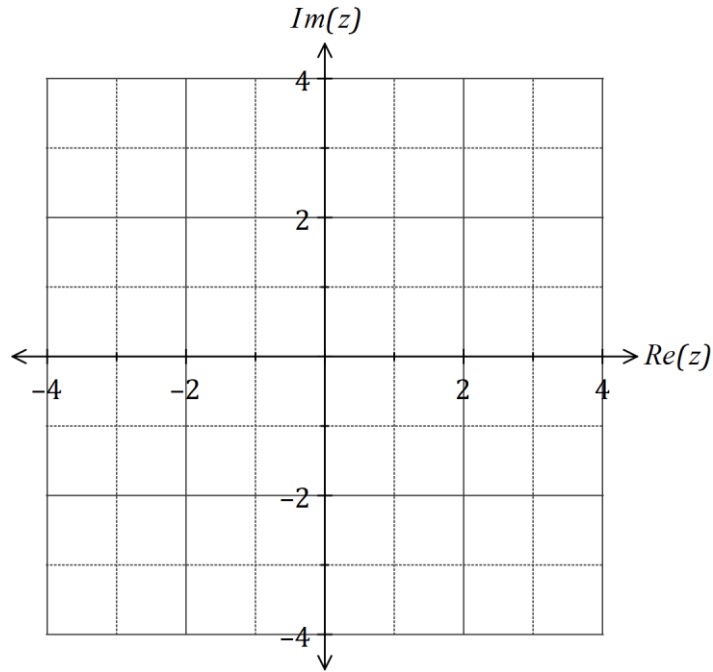
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Question 21 (6 marks)

Sketch the locus of the complex number z given by

(a) $|z + 3| \leq |z - 1 + 2i|$. (3 marks)
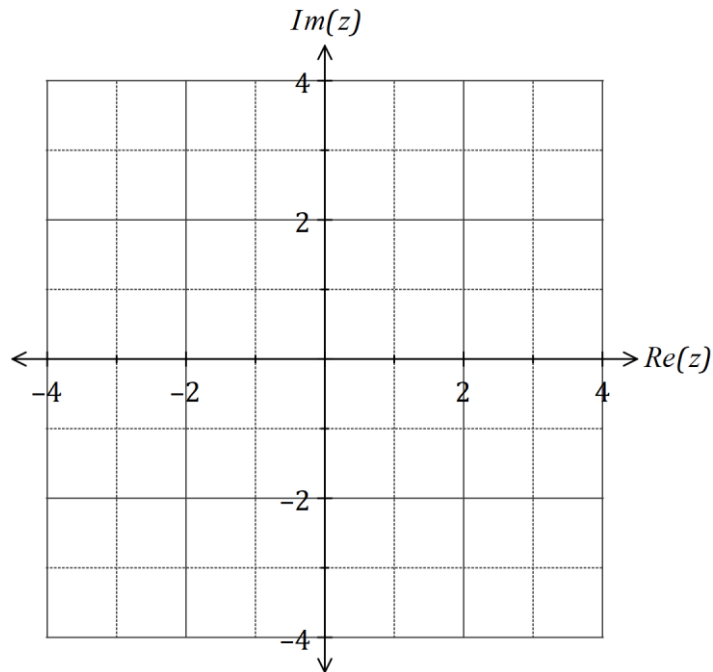

(b) $|z + 2| = |z| - 2$. (3 marks)

End of questions

Supplementary page

Question number: _____